

The elliptic Weyl character formula

Nora Ganter*

The University of Melbourne

January 26, 2013

Abstract

We calculate equivariant elliptic cohomology of the partial flag variety G/H , where $H \subseteq G$ are compact connected Lie groups of equal rank. We identify the $RO(G)$ -graded coefficients $\mathcal{E}ll_G^*$ as powers of Looijenga's line bundle and prove that transfer along the map

$$\pi: G/H \longrightarrow \text{pt}$$

is calculated by the Weyl-Kac character formula. Treating ordinary cohomology, K -theory and elliptic cohomology in parallel, this paper organizes the theoretical framework for the elliptic Schubert calculus of [GR].

1 Introduction

The topological aspects of representation theory are captured by the generalized cohomology theory known as equivariant K -theory. Applied to a point, the K -group

$$K_G(\text{pt}) = R(G)$$

is the representation ring of the structure group G . Applied to other spaces, it yields rings, which are related to the representation rings. For instance,

*This work was made possible by a Centenary Fellowship from the Faculty of Science at the University of Melbourne and by an Australian Research Fellowship.

the K_G -theoretic transfer along the map $\pi: G/H \rightarrow \text{pt}$ gives the induction map

$$\text{ind}: R(H) \longrightarrow R(G).$$

This point of view is taken in [AB68], where Atiyah and Bott obtain Weyl's famous formula for the character of $\text{ind}([\varrho])$ as an application of their fixed point formula for the T -equivariant transfer $\pi_!$. Here T is a maximal torus, sitting inside H .

Schubert calculus, originally concerned with the cohomology of the partial flag varieties, has long been extended to include the analogous K -theory picture. An essential ingredient in both theories are pull-backs and transfers (push-forwards) along maps between partial flag varieties.

In [BE90], Bressler and Evens formulate Schubert calculus in broad generality, replacing cohomology and K -theory with any generalized multiplicative cohomology theory possessing the relevant transfers. The universal example of such a theory is complex cobordism, and cobordism-theoretic Schubert calculus is now becoming a discipline of its own (see [BE92], [HK11] and [CPZ09]).

We are interested in equivariant elliptic cohomology, $\mathcal{E}ll_G$. It has long been conjectured that $\mathcal{E}ll_G$ plays the same role for the representation theory of the loop group $\mathcal{L}G$ that K_G plays for the representation theory of G . This idea can already be found in Grojnowski's article on the definition of $\mathcal{E}ll_G$ [Gro07, p.2 and 3.3], it took shape in Ando's work on Euler classes [And03] and was later picked up by Lurie [Lur]. We will see in Section 6.2 that

$$\Gamma \mathcal{E}ll_G^*(\text{pt}) \cong \widetilde{Th}_*^W$$

is Looijenga's ring of theta functions [Loo77]. This is where the loop group characters take their values.¹

The paper at hand is the first in a joint program with Arun Ram, studying Schubert calculus in elliptic cohomology. For this, the ring \widetilde{Th}_* will play the same role as $R(T)$ for the K -theoretic Schubert calculus or as the symmetric algebra $S(\mathfrak{t}_{\mathbb{C}}^*)$ in cohomology. Our work ties in with the Bressler-Evens program, but is not a special case of [BE90]: Bressler and Evens work Borel equivariantly, while Grojnowski's $\mathcal{E}ll_T$ is a genuinely equivariant theory, taking values in sheaves over a scheme \mathcal{M}_T . More importantly, $\mathcal{E}ll_T$ does not

¹More precisely, we are considering characters of positive energy representations of the central extension $\widehat{\mathcal{L}G}$.

possess Thom isomorphisms for complex vector bundles, so that the theory of transfer maps acquires a twist by a line bundle, called the *Thom sheaf*.

In our main application, the Thom sheaf for $\pi_!$ turns out to be \mathcal{L}_{Lo}^g , the Looijenga line bundle raised to the dual Coxeter number. This accounts for the shift of level by g occurring in the Weyl-Kac formula.

The paper is organized as follows: treating cohomology, K -theory and elliptic cohomology simultaneously, we view all three theories as sheaf valued, revisiting, and to some extent reorganizing, the circle of ideas in [Gro07], [Ro03], [Ro01], [And03], [GKV95], [Lur], and [Gep05].

After recalling the definitions and the general setup (Sections 2 and 3), we review a powerful calculational tool: this is the theory of moment graphs (Section 4). We show how to deduce the isomorphism

$$K_T(G/H) \cong R(T) \otimes_{R(G)} R(H)$$

for torsion free $\pi_1 G$ (see [McL79]) directly from the moment graph of G/H . Our proof does not use Pittie's theorem that $R(H)$ is free over $R(G)$, nor does it involve any explicit calculations with basis elements. Our argument is identical for K -theory and cohomology and also yields a description of $\mathcal{E}ll_T(G/H)$.

Now we are in a position to prove the axioms of [GKV95] needed in our applications, and this is done in Section 5.

Section 6 treats the theory of Thom sheaves. We identify The Thom sheaf \mathbb{L}^ξ of any complex vector bundle ξ with the pull-back of a universal example:

$$\mathbb{L}^\xi \cong c_\xi^* \mathbb{L}^{univ}.$$

Here c_ξ is the Ginzburg-Kapranov-Vasserot characteristic class of ξ .

Finally, in Section 8, we arrive at the promised formula for the transfer $\pi_!$. In cohomology, this is a formula by Akyildiz and Carrell [AC83], in K -theory, it is the Weyl formula, and in elliptic cohomology it is the Weyl-Kac formula.

The combinatorial aspects of the theory, as well as concrete examples will be addressed in [GR].

1.1 Acknowledgments

I would like to thank Matthew Ando and Ioanid Roşu for teaching me much of what I know about elliptic cohomology. Many thanks go to Craig Westerland

and Alex Ghitza for countless conversations. This paper is my account of work that is, to a large extent, joint with Arun Ram. It is a pleasure to thank him for being such an inspiring collaborator. Finally, I would like to take this opportunity to express my deep gratitude to Arun, my department and the Faculty of Science for making it possible to continue this work in spite of very difficult circumstances.

Contents

1	Introduction	1
1.1	Acknowledgments	3
2	The three sheaf valued theories	5
2.1	Homogeneous spaces and representation spheres	7
2.2	Stalks	8
3	The Chern character and the construction of $\mathcal{E}ll_T$	9
3.1	Completion	9
3.2	Roşu's Chern character	10
3.3	Construction of $\mathcal{E}ll_T$	11
3.4	Compact connected Lie groups	13
4	Moment graphs	13
4.1	The scheme X_{A_G}	15
4.2	Partial flag varieties	16
5	The Ginzburg-Kapranov-Vasserot characteristic class	19
5.1	Properties of X_{A_G}	19
5.2	Classifying maps	22
6	Thom sheaves	24
6.1	Properties of the Thom sheaf	25
6.2	$RO(G)$ -grading and periodicity	30
7	Euler classes	31
7.1	Thom isomorphisms	31
7.2	Theta functions and elliptic Euler classes	33
7.3	Push-forwards	34

8	Character Formulas	36
8.1	Induced representations	36
8.2	The Weyl character formula	37
8.3	The Kac character formula	39

2 The three sheaf valued theories

Let G be a compact Lie group, and let X be a finite G -CW-complex. We write

$$H_G(X) := \sum_{n \in \mathbb{Z}} H^{2n}(EG \times_G X; \mathbb{C})$$

for (even) Borel equivariant cohomology with complex coefficients, and

$$K_G(X) := (\text{Vect}_G^{\mathbb{C}}(X))^{\text{gp}} \otimes_{\mathbb{Z}} \mathbb{C}$$

for equivariant K -theory (as in [Seg68]) with complex coefficients. We will also consider the relative and reduced versions of these theories. These are contravariant functors in G and in X (or pairs (X, A) or (X, x_0)). For abelian T it follows that the coefficient rings

$$H_T := H_T(\text{pt}) \quad \text{and} \quad K_T := K_T(\text{pt})$$

form Hopf algebras, with the comultiplication given by multiplication in T . For the circle group $U(1)$, we have

$$\begin{aligned} H_{U(1)} &= \mathbb{C}[x] \quad \text{and} \\ K_{U(1)} &= \mathbb{C}[z^{\pm 1}]. \end{aligned}$$

These are the Hopf algebras of regular functions on the (affine) group schemes

$$\begin{aligned} \mathbb{G}_a &= \mathbb{A}_{\mathbb{C}}^1 && \text{(additive group) and} \\ \mathbb{G}_m &= \mathbb{A}_{\mathbb{C}}^1 \setminus \{0\} && \text{(multiplicative group).} \end{aligned}$$

Here

$$x = c_1(\mathbb{C}_1)_{U(1)}$$

is the first Borel equivariant Chern class of the defining representation \mathbb{C}_1 of $U(1)$. It generates the ideal

$$x\mathbb{C}[x] = I(0)$$

of regular functions on $\mathbb{A}_{\mathbb{C}}^1$ that vanish at 0. The K -theory class z is the character of \mathbb{C}_1 . The $K_{U(1)}$ -theoretic first Chern class of \mathbb{C}_1 equals $1 - z$, generating the ideal $I(1)$ of regular functions on \mathbb{G}_m vanishing at 1.

More generally, let T be a compact abelian Lie group with Lie algebra \mathfrak{t} , and let

$$\widehat{T} = \text{Hom}(T, U(1))$$

be the character lattice of T . Let $T_0 \subseteq T$ be the connected component of 1, and let $\Lambda \subset \mathfrak{t}^*$ be the weight lattice of the torus T_0 . If $T = T_0$ is connected there is an isomorphism

$$\begin{aligned} \Lambda &\xrightarrow{\cong} \widehat{T} \\ \lambda &\longmapsto e^{2\pi i \lambda}. \end{aligned}$$

For $\lambda \in \Lambda$, let \mathbb{C}_λ be the one dimensional representation of T_0 with character $e^{2\pi i \lambda}$. Then the coefficients $H_T \cong H_{T_0}$ are identified by the Hopf-algebra isomorphism

$$\begin{aligned} H_T &\cong \Gamma \mathcal{O}_{\mathfrak{t}_{\mathbb{C}}} \\ c_1(\mathbb{C}_\lambda)_{T_0} &\longleftarrow \lambda_{\mathbb{C}}. \end{aligned}$$

Here $\lambda_{\mathbb{C}} = \lambda \otimes_{\mathbb{R}} \mathbb{C}$ is viewed as a regular function on the complex algebraic group $\mathfrak{t}_{\mathbb{C}} := \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$. This point of view, going back to Borel, allows us to interpret $H_T(X, A)$ as the global sections of a coherent sheaf $\mathcal{H}_T(X, A)$ on $\mathfrak{t}_{\mathbb{C}}$.

In K -theory, the T -equivariant coefficients are given by the representation ring

$$K_T = R(T) = \mathbb{C}[\widehat{T}].$$

For instance,

$$K_{T_0}(\text{pt}) \cong \mathbb{C}\{e^\lambda\}_{\lambda \in 2\pi i \Lambda},$$

with $e^\lambda e^\mu = e^{\lambda+\mu}$. So,

$$K_T \cong \Gamma \mathcal{O}_{T_{\mathbb{C}}}$$

is identified with the ring of regular functions of the complexification $T_{\mathbb{C}}$ of T . This allows us to view $K_T(X, A)$ as the global sections of a coherent sheaf $\mathcal{K}_T(X, A)$ on $T_{\mathbb{C}}$.

We have

$$\begin{aligned} T_{\mathbb{C}} &\cong \operatorname{Hom}(\widehat{T}, \mathbb{C}^{\times}), \quad \text{and} \\ \mathfrak{t}_{\mathbb{C}} &\cong \operatorname{Hom}(\widehat{T}, \mathbb{C}). \end{aligned}$$

Let E be a complex elliptic curve, and let²

$$\mathcal{M}_T := \operatorname{Hom}(\widehat{T}, E).$$

Grojnowski's T -equivariant elliptic cohomology takes values in coherent sheaves over \mathcal{M}_T , and

$$\mathcal{E}ll_T(\text{pt}) = \mathcal{O}_{\mathcal{M}_T}.$$

We will see that the above theories form the degree zero parts of three $RO(T)$ -graded (sheaf valued) equivariant cohomology theories. Note that the complex group \mathcal{M}_T is no longer affine and the global sections $\Gamma \mathcal{E}ll_T(-)$ do not form a cohomology theory. This makes the sheaf point of view essential to the theory. We will now see how the formal properties of $\mathcal{E}ll_T$, as axiomatically postulated in [GKV95], determine the stalks of the theory. This is the motivation behind Grojnowski's construction, which we will recall in Section 3.3.

2.1 Homogeneous spaces and representation spheres

Let $T' \subseteq T$ be a closed subgroup. Then we have canonical inclusions $\mathfrak{t}'_{\mathbb{C}} \subseteq \mathfrak{t}_{\mathbb{C}}$ and $T'_{\mathbb{C}} \subseteq T_{\mathbb{C}}$ and $\mathcal{M}_{T'} \subseteq \mathcal{M}_T$ and isomorphisms of coherent sheaves

$$\begin{aligned} \mathcal{H}_T(T/T') &\cong \mathcal{O}_{\mathfrak{t}'_{\mathbb{C}}} \quad (\text{over } \mathfrak{t}_{\mathbb{C}}) \\ \mathcal{K}_T(T/T') &\cong \mathcal{O}_{T'_{\mathbb{C}}} \quad (\text{over } T_{\mathbb{C}}) \\ \mathcal{E}ll_T(T/T') &\cong \mathcal{O}_{\mathcal{M}_{T'}} \quad (\text{over } \mathcal{M}_T). \end{aligned} \tag{1}$$

The first two of these isomorphisms are classical. We recall the definition of the third on page 20. That it is an isomorphism will be an immediate consequence of the construction of $\mathcal{E}ll_T$.

From now on, we let A_T be one of the complex abelian groups $\mathfrak{t}_{\mathbb{C}}$ or $T_{\mathbb{C}}$ or \mathcal{M}_T , and we let \mathcal{F}_T be the theory \mathcal{H}_T or \mathcal{K}_T or $\mathcal{E}ll_T$ taking values in sheaves

² \mathcal{M}_T may be interpreted as the moduli scheme of certain principal T -bundles on E , see [GKV95, (1.4.2)], where \mathcal{M}_G is denoted \mathcal{X}_G .

over A_T . Often we will write $+$ for the group operation in A_T and 0 for its unit, with the understanding that these are to be replaced by \cdot and 1 for the multiplicative case $A_T = T_{\mathbb{C}}$.

Let T be a torus, $\lambda\Lambda$, and let SS^λ be the representation sphere (one point compactification) of \mathbb{C}_λ , and write K_λ for the kernel of $e^{2\pi i\lambda}$ inside T . We may identify the equator of SS^λ with T/K_λ . The usual Mayer-Vietoris argument gives the following:

Corollary 2.1. *The sheaf $\mathcal{F}_T(SS^\lambda)$ is identified with the kernel of the map*

$$\begin{aligned} \mathcal{O}_{A_T} \oplus \mathcal{O}_{A_T} &\longrightarrow \mathcal{O}_{A_{K_\lambda}} \\ (f, g) &\longmapsto (f - g)|_{A_{K_\lambda}}. \end{aligned}$$

2.2 Stalks

By a point in A_T , we will always mean a maximal point. For $a \in A_T$, let

$$T(a) := \bigcap_{a \in A_{T'}} T'$$

be the smallest subgroup of T with $a \in A_{T(a)}$. Let

$$i_a: X^{T(a)} \hookrightarrow X$$

be the inclusion of the $T(a)$ -fixed points. We will identify the stalk of \mathcal{F}_T at a in two steps.

First, we note that

$$i_a^*: \mathcal{F}_T^*(X)_a \xrightarrow{\cong} \mathcal{F}_T^*(X^{T(a)})_a \quad (2)$$

is an isomorphism of T -equivariant cohomology theories. Indeed, it is enough to check this on orbits $X = T/T'$, where it follows from (1). Second, consider the quotient map $p: T \rightarrow T/T(a)$ and use the isomorphism³

$$\mathcal{F}_T(X^{T(a)}) \cong A_p^*(\mathcal{F}_{T/T'}(X^{T(a)})).$$

Let

$$\begin{aligned} \tau_a: A_T &\longrightarrow A_T \\ b &\longmapsto a + b \end{aligned}$$

³This isomorphism was proved in [AB84] for H , in [Seg68] for K and postulated for $\mathcal{E}l$ in [GKV95, (1.6.3)]. Again, it will follow immediately from the construction of $\mathcal{E}l_T$.

denote translation by a . Then

$$A_p = A_p \circ \tau_a, \quad (3)$$

and hence

$$\mathcal{F}_T(X^{T(a)})_a \cong \mathcal{F}_T(X^{T(a)})_0.$$

Combining these two steps, we obtain isomorphisms

$$\begin{aligned} \mathcal{H}_T(X)_a &\cong H_T(X^{T(a)}) \otimes_{H_T} \mathcal{O}_{\mathfrak{t}_{\mathbb{C}},0} \\ \mathcal{K}_T(X)_a &\cong K_T(X^{T(a)}) \otimes_{K_T} \mathcal{O}_{T_{\mathbb{C}},1} \quad \text{and} \\ \mathcal{E}ll_T(X)_a &\cong \mathcal{E}ll_T(X^{T(a)})_0. \end{aligned}$$

Over a sufficiently small neighbourhood U of a (see Section 3.3 for details) these isomorphisms extend to an isomorphism of sheaves

$$\mathcal{F}_T(X)|_U \cong (\tau_a)_* (\mathcal{F}_T(X^{T(a)})|_{U-a}). \quad (4)$$

3 The Chern character and the construction of $\mathcal{E}ll_T$

3.1 Completion

In the sheaf-theoretic language, the Atiyah-Segal completion theorem identifies the formal completion of \mathcal{F}_T at $0 \in A_T$ with the Borel equivariant version of \mathcal{F} . More precisely, we have the following theorem.

Theorem 3.1 (Completion Theorem). *We have an isomorphism of pro-rings*

$$\mathcal{F}_T(X)_0^\wedge \cong \varprojlim_k \mathcal{F}(ET^{(k)} \times_T X),$$

where $ET^{(k)}$ is the k -skeleton of ET .

In the case of K -theory, the right-hand side is $K(ET \times_T X)$, and Theorem 3.1 is [AS69]. For cohomology, the right-hand side is

$$\prod_{n \in \mathbb{Z}} H^{2n}(ET \times_T X; \mathbb{C})$$

(see [Ro03, p.6]). In Section 6, we will see how Theorem 3.1 follows from the formal properties of \mathcal{F}_T .

3.2 Roşu's Chern character

Consider the exponential map

$$\exp: \mathfrak{t}_{\mathbb{C}} \longrightarrow T_{\mathbb{C}}.$$

This is an analytic map of complex groups, it is not algebraic. For an algebraic sheaf \mathcal{F} on a complex variety, we let \mathcal{F}^h be the analytic sheaf associated to \mathcal{F} . The following theorem is a reformulation of the main result in [Ro03].

Theorem 3.2 (Roşu). *Assume⁴ that $H_T(X)$ is free over H_T . For a small enough analytic open neighbourhood U of $0 \in \mathfrak{t}_{\mathbb{C}}$ there is an isomorphism of analytic sheaves*

$$ch_T: \mathcal{K}_T^h(X)|_{\exp(U)} \xrightarrow{\cong} \exp_* \mathcal{H}_T^h(X)|_U,$$

uniquely determined by the commuting diagram

$$\begin{array}{ccc} \mathcal{K}_T^h(X)_1 & \xrightarrow{ch_T|_1} & \mathcal{H}_T^h(X)_0 \\ \downarrow & & \downarrow \\ \mathcal{K}_T(X)_1^{\widehat{}} & \xrightarrow{ch} & \mathcal{H}_T(X)_0^{\widehat{}}, \end{array}$$

where the top row is Roşu's Chern character at the stalk 1, and in the bottom row, ch stands for the (Borel-equivariant) classical Chern character.

Consider now the quotient maps

$$\exp_E: \mathbb{C} \longrightarrow E = \mathbb{C}/2\pi i\langle \tau, 1 \rangle$$

and

$$y: \mathbb{C}^{\times} \longrightarrow E \cong \mathbb{C}^{\times}/q^{\mathbb{Z}},$$

⁴This assumption ensures that the restriction maps of $\mathcal{H}_T^h(X)$ and, more importantly, the map $\mathcal{H}_T^h(X)_0 \rightarrow \mathcal{H}_T(X)_0^{\widehat{}}$ are injective. I do not follow Roşu's argument for arbitrary X in [Ro03, p.7]. This does not affect the main applications in [Ro03], since for those Knutsen and Roşu do require $H_T(X)$ to be free over H_T .

where $q = e^{2\pi i\tau}$. These induce a commuting diagram of complex analytic group homomorphisms

$$\begin{array}{ccc} \mathfrak{t}_{\mathbb{C}} & \xrightarrow{\exp_{T_{\mathbb{C}}}} & T_{\mathbb{C}} \\ & \searrow \exp_{\mathcal{M}_T} & \swarrow y \\ & \mathcal{M}_T & \end{array}$$

When it exists, Roşu's Chern isomorphism ch_T will fit into a commuting diagram

$$\begin{array}{ccc} (\exp_{\mathcal{M}_T})_* \mathcal{H}_T^h(X)|_U & \xleftarrow{y_*(ch_T)} & y_* \mathcal{K}_T^h(X)|_{\exp_{T_{\mathbb{C}}}(U)} \\ & \swarrow \phi & \searrow \psi \\ & \mathcal{E}ll_T^h(X)|_{\exp_{\mathcal{M}_T}(U)} & \end{array}$$

of sheaf isomorphisms over a small neighbourhood of 0 in \mathcal{M}_T .

3.3 Construction of $\mathcal{E}ll_T$

The construction of $\mathcal{E}ll_T(X)$ was first outlined in [Gro07]. The technical details were filled in in [Ro03], see also [And03] and [Ro01]. This section is a reminder of Grojnowski's construction.

Note first that the properties stated in Section 2.2 and Section 3.2 determine $\mathcal{E}ll_T^h(X)$ locally: every $a \in \mathcal{M}_T$ has a small analytic neighbourhood U_a satisfying

$$b \in U_a \implies X^{T(b)} \subseteq X^{T(a)}.$$

Choose U_0 small enough such that the logarithm is well-defined over it and

$$c \in U_0 \implies X^{T_{\log(c)}} = X^{T(c)}.$$

Further, we assume U_a to be small enough to satisfy

$$(U_a - a) \subseteq U_0.$$

Then we are forced into

$$\mathcal{E}ll_T^h(X)|_{U_a} \cong (\tau_a \circ \exp)_* \mathcal{H}_T^h(X^{T(a)})|_{\log(U_a - a)}.$$

We need to understand how these patches are to be glued. Given a non-empty intersection $U := U_a \cap U_b$, we make the additional assumptions⁵

$$\begin{aligned} a - b &\in U_0, & \text{and} \\ X^{T(b)} &\subseteq X^{T(a)}. \end{aligned}$$

Lemma 3.3. *Let i denote the inclusion of $X^{T(b)}$ in $X^{T(a)}$. After restricting to $\log(U - a)$, the map*

$$i^*: \mathcal{H}_T(X^{T(a)}) \xrightarrow{\cong} \mathcal{H}_T(X^{T(b)})$$

becomes an isomorphism of (analytic) sheaves.

PROOF : We check the statement on stalks. Let $\gamma \in \log(U - a)$. We claim that we have an equality of simultaneous fixed point sets

$$X^{T(a)} \cap X^{T(\gamma)} = X^{T(b)} \cap X^{T(\gamma)}.$$

The statement then follows from (2) with γ in the role of a . To prove the non-trivial direction of the claim, let $c = \exp(\gamma)$. Then $a + c$ is an element of U_b , and we obtain

$$X^{T(a)} \cap X^{T(\gamma)} = X^{T(a)} \cap X^{T(c)} \subseteq X^{T(a+c)} \subseteq X^{T(b)}.$$

The inclusion in the middle may be checked on orbits $T/T' \subseteq X$, where it follows immediately from the definition of $T(-)$. \square

Let now $\gamma := \log(a - b)$. Similarly to the proof of the lemma, one argues that T_γ fixes $X^{T(b)}$. As in (3), we obtain an isomorphism

$$\phi: (\tau_\gamma)_* \mathcal{H}_T(X^{T(b)}) \cong \mathcal{H}_T(X^{T(b)}),$$

and hence of the corresponding analytic sheaves. Finally, we have

$$\tau_b \circ \exp \circ \tau_\gamma = \tau_a \circ \exp.$$

The desired glueing isomorphism is the composite

$$(\tau_b \exp)_* (\phi) \circ (\tau_a \exp)_* (i^*).$$

To define the algebraic theory $\mathcal{E}ll_T(-)$, we use Serre's GAGA result:

⁵This is possible by [Ro03, 2.5].

Theorem 3.4 ([Ser56]). *Let X be a projective algebraic variety over \mathbb{C} , let X^h be its underlying analytic variety and let $\mathcal{C}oh_{\text{alg}}(X)$ and $\mathcal{C}oh_{\text{an}}(X^h)$ be the categories of coherent algebraic (resp. analytic) sheaves over X . Then the functor*

$$\begin{aligned} \mathcal{C}oh_{\text{alg}}(X) &\longrightarrow \mathcal{C}oh_{\text{an}}(X^h) \\ \mathcal{F} &\longmapsto \mathcal{F}^h \end{aligned}$$

is an equivalence of categories.

3.4 Compact connected Lie groups

Let G be a compact connected Lie group with maximal torus T and Weyl group W . Then \mathcal{F}_G takes values over the scheme

$$A_G = A_T/W,$$

and

$$\mathcal{F}_G(X) = \mathcal{F}_T(X)^W$$

is the sheaf of W -invariant sections. In the elliptic case, these are to be taken as the definition of \mathcal{M}_G and $\mathcal{E}ll_G$.

4 Moment graphs

Moment graph theory provides a powerful tool for calculations. Let T be a compact torus and X a compact T -manifold. Let

$$i: X^T \longrightarrow X$$

be the inclusion of the fixed points in X . For a subtorus T' of T , this factors through the inclusion

$$i_{T'}: X^T \longrightarrow X^{T'}.$$

Recall that the equivariant 1-skeleton X_1 of X is defined as the set of all points in X whose orbit is at most one-dimensional. The following theorem was proved for cohomology by Goresky, Kottwitz and MacPherson [GKM98]. Knutsen and Roşu later generalized it to K -theory and elliptic cohomology [Ro03].

Theorem 4.1 (Localization Theorem). *Assume that $H_T(X)$ is free over H_T and that X_1 consists of a finite number of representation spheres SS^λ , meeting only at the fixed points. Then the map*

$$i^*: \mathcal{F}_T(X) \longrightarrow \mathcal{F}_T(X^T)$$

is injective, and its image is equal to

$$\mathrm{Im}(i^*) = \bigcap_{T'} \mathrm{Im}(i_{T'}^*), \quad (5)$$

where the intersection runs over all subgroups of codimension 1 in T .

The data determining the right-hand side of (5) are recorded in the “moment graph” of X :

Definition 4.2. In the situation of the theorem, the *moment graph* Γ of X has vertices indexed by the fixed points of X and an oriented edge with label $\lambda \in \Lambda$ from x_1 to x_2 for each $SS^\lambda \subseteq X_1$, containing x_1 as 0 and x_2 as ∞ .

Corollary 4.3 (of Theorem 4.1 and Corollary 2.1). *In the situation of the theorem, $\mathcal{F}_T(X)$ is described by the following equalizer diagram*

$$\mathcal{F}_T(X) \longrightarrow \bigoplus_v \mathcal{O}_{A_T} \rightrightarrows \bigoplus_{(e,\lambda)} \mathcal{O}_{A_{K_\lambda}}.$$

Here $K_\lambda = \ker(e^{2\pi i\lambda})$, the first sum is over the vertices, the second sum is over the edges of the moment graph, and the two arrows are defined in the obvious manner.

This formulation of the theory can be found in the paragraph before (1.3) in [GKM98, p.27].

Example 4.4 (Partial flag varieties). Let $H \subseteq G$ be compact connected Lie groups of equal rank. Let $T \subseteq H$ be a maximal torus (of both). Let W_G be the Weyl group of G . Then the Weyl group of H can be identified with a subgroup $W_H \subseteq W_G$. Fix a set of positive roots \mathcal{R}_+ of G . For $\alpha \in \mathcal{R}_+$, let $s_\alpha \in W_G$ be the corresponding reflection. The following description of the moment graph of G/H can be found in [Tym09, Thm 3.1]: we have a bijection

$$\begin{aligned} W_G/W_H &\longrightarrow (G/H)^T \\ wW_H &\longmapsto wH. \end{aligned}$$

Writing $[w]$ for the vertex corresponding to the left-coset wW_H , we have an edge labeled α from $[w]$ to $[s_\alpha w]$ whenever $\alpha \in \mathcal{R}_+$ is such that $w^{-1}(\alpha) < 0$ is not a root of H .

Often the groups $K_\alpha = \ker(e^{2\pi i\alpha})$, turning up as the stabilizers of one dimensional orbits in G/H , have an interpretation as fixed points:

Lemma 4.5. *Let G be a compact connected Lie group with maximal torus T and Weyl group W . Let α be a root of G . Then the action of s_α on T leaves the elements of K_α fixed. If $\pi_1(G)$ is torsion free then the inclusion*

$$K_\alpha \subseteq T^{s_\alpha}$$

is an equality.

PROOF : The first claim is [BtD85, V.(2.9)(iii)]. Recall from [BtD85, V.(7.1)] that

$$\pi_1(G) = \Lambda^\vee / \Gamma,$$

where

$$\Lambda^\vee = \ker(\exp) \subseteq \mathfrak{t}$$

and Γ is the sublattice generated by the coroots. Let $x \in \mathfrak{t}$ be such that $\exp(x)$ is fixed under s_α . Then

$$\alpha(x)\check{\alpha} = x - s_\alpha(x)$$

is an element of Λ^\vee . Since Λ^\vee / Γ is torsion free, it follows that $\alpha(x)$ is an integer. Hence $e^{2\pi i\alpha(x)} = 1$, and we have proved

$$\exp(x) \in K_\alpha.$$

□

4.1 The scheme X_{A_G}

The sheaf $\mathcal{F}_G(X)$ is a sheaf of commutative algebras over A_G . Following [GKV95, (1.7.4)], we let X_{A_G} be the spectrum of $\mathcal{F}_G(X)$. This is a scheme over A_G . The assignment

$$X \longmapsto X_{A_G}$$

is covariantly functorial in X . We have $\text{pt}_{A_G} = A_G$. Writing $\pi: X \rightarrow \text{pt}$ for the unique map from X to the one point space, the map

$$\pi_{A_G}: X_{A_G} \longrightarrow A_G$$

is the structure morphism. In other words, X_{A_G} is determined by the fact that

$$(\pi_{A_G})_* \mathcal{O}_{X_{A_G}} \cong \mathcal{F}_G(X).$$

4.2 Partial flag varieties

Let $H \subseteq G$ be compact, connected Lie groups of equal rank, let $T \subseteq H$ be a maximal torus, W_G the Weyl group of T in G .

Theorem 4.6. *Assume that $\pi_1(G)$ is torsion free. Then we have a W_G -equivariant epimorphism of schemes over A_T*

$$\varphi: (G/H)_{A_T} \longrightarrow A_T \times_{A_G} A_H,$$

inducing an isomorphism of schemes over A_G

$$(G/H)_{A_G} \cong A_H.$$

In the case of ordinary cohomology, the assumption that $\pi_1(G)$ is torsion free is not needed.

PROOF OF THEOREM 4.6: With the notation as in Example 4.4, we write

$$F := W_G/W_H$$

for the set of vertices in the moment graph. Consider the map

$$\begin{aligned} \overline{\varphi}: \coprod_{[w] \in F} A_T &\longrightarrow A_T \times_{A_G} A_H \\ ([w], a) &\longmapsto (a, [w^{-1}a]). \end{aligned}$$

Let W_G act on the source of $\overline{\varphi}$ by

$$v \cdot ([w], a) = ([vw], va),$$

and on the target by its usual action on the first factor. Then $\overline{\varphi}$ is W_G -equivariant. By Corollary 4.3 and Example 4.4, we have a coequalizer diagram

$$\begin{array}{ccc} \coprod_{[w], \alpha} A_{K_\alpha} & \rightrightarrows & \coprod_{[w] \in F} A_T \longrightarrow (G/H)_{A_T} \\ ([w], \alpha, a) & \longmapsto & ([w], a) \\ ([w], \alpha, a) & \longmapsto & ([s_\alpha w], a). \end{array}$$

where the first coproduct runs over the edges of the moment graph. Both maps on the left are W_G -equivariant with respect to the action

$$v \cdot ([w], \alpha, a) = ([vw], v(\alpha), va)$$

on their source. The universal property of coequalizer yields a W_G -equivariant map

$$\varphi: (G/H)_{A_T} \longrightarrow A_T \times_{A_G} A_H,$$

which is easily seen to be an epimorphism. Note that we have identified $\overline{\varphi}$ with A_i , where i is the inclusion of the T -fixed points F in G/H . To obtain the promised map of schemes over A_G , we quotient by the action of W_G . It remains to prove injectivity of φ/W_G . We have

$$\overline{\varphi}([w], a) = \overline{\varphi}([v], b) \iff a = b \quad \text{and} \quad [w^{-1}a] = [v^{-1}a].$$

In the source of φ , we have made the, a priori finer, identifications

$$([w], a) \sim ([s_\alpha w], a) : \iff s_\alpha a = a. \tag{6}$$

Here we have used Lemma 4.5, which is why we need the assumption that $\pi_1(G)$ be torsion free. In many cases (6) is sufficient to imply injectivity of φ , but we will see an example where this fails. Assume now that $[w^{-1}a] = [v^{-1}a]$. Then there is an element $u \in W_H$ with

$$w^{-1}a = uv^{-1}a.$$

In $(G/H)_{A_G}$ we have

$$([w], a) \sim ([1], w^{-1}a) = ([1], uv^{-1}a) \sim ([vu^{-1}], a) = ([v], a).$$

Hence φ/W_G is an isomorphism, as claimed. \square

We now ask when the map φ of the Theorem is injective.

Lemma 4.7. *Let $w \in W_G$ and let T_c^w be a connected component of the subgroup of w -fixed points in T . Then we can write w as a word*

$$w = s_{\alpha_1} \cdots s_{\alpha_l}$$

in (not necessarily simple) reflections such that

$$T_c^w \subseteq T^{s_{\alpha_1}} \cap \cdots \cap T^{s_{\alpha_l}}.$$

PROOF : Choose $t \in T_c^w$ with $T_c^w \subseteq \overline{\langle t \rangle}$. Let $Z_G(t)$ be the centralizer of t in G . This is a connected closed subgroup of full rank. Its Weyl group W_Z may be viewed as a reflection subgroup of W_G . All elements of W_Z fix t and hence T_c^w . Since $w \in W_Z$, we can write w as a word in the reflections generating W_Z . \square

The following example shows that the s_{α_j} in Lemma 4.7 can not always be chosen independently of the connected component.

Example 4.8. Let $G = G_2$, and consider the element $w \in W$ acting by $(-)^{-1}$ on T . Then $T^w = T[2]$ has four elements. For each non-trivial element of $t \in T^w$ there is a different, unique pair of reflections $s_{\alpha_t}, s_{\beta_t} \in W$ fixing t . For each such pair, $w = s_{\alpha_t} s_{\beta_t}$.

Corollary 4.9 (of Lemma 4.7). *In the situation of the lemma, we have*

$$\mathfrak{t}_{\mathbb{C}}^w = \mathfrak{t}_{\mathbb{C}}^{s_{\alpha_1}} \cap \cdots \cap \mathfrak{t}_{\mathbb{C}}^{s_{\alpha_l}}$$

and

$$(T_{\mathbb{C}}^w)_c \subseteq T_{\mathbb{C}}^{s_{\alpha_1}} \cap \cdots \cap T_{\mathbb{C}}^{s_{\alpha_l}}.$$

Corollary 4.10. *For cohomology and K -theory, the map φ of Theorem 4.6 is an isomorphism. If the centralizers of commuting pairs in G are connected, then φ is also an isomorphism in elliptic cohomology.*

On global sections, φ gives the familiar isomorphisms

$$H_T \otimes_{H_G} H_H \cong H_T(G/H),$$

studied by Borel, Demazure and others, and

$$R(T) \otimes_{R(G)} R(H) \cong K_T(G/H)$$

[McL79].

Remark 4.11. The condition that the centralizers of commuting pairs be connected, should be compared to [Gro07, 3.2].

Example 4.12. Let $G = U(n)$. Then all the groups T^w are, in fact, connected, so that the word in Lemma 4.7 depends only on w , so that the map φ of the theorem is an isomorphism also in the elliptic case.

5 The Ginzburg-Kapranov-Vasserot characteristic class

5.1 Properties of X_{A_G}

We discuss some basic properties of the scheme X_{A_G} introduced in Section 4.1. In the cases of cohomology and K -theory, these are well-known. In the elliptic case, they were conjectured in [GKV95]. We give proofs for the special cases relevant to us.

Change of groups: Let $\phi: H \rightarrow G$ be a map of groups, and let X be a finite G -CW-complex. Then we have a commuting square

$$\begin{array}{ccc} X_{A_H} & \longrightarrow & X_{A_G} \\ \pi_{A_H} \downarrow & & \downarrow \pi_{A_G} \\ A_H & \xrightarrow{A_\phi} & A_G, \end{array}$$

We write X_{A_ϕ} for the top map. The assignment $(-)_{A_\phi}$ is natural in X .

Induction Axiom: Let $K \triangleleft G$ be a normal subgroup, and let X be a G -space such that the action of K on X is free. Write $p: X \rightarrow K \backslash X$ and $\phi: G \rightarrow G/K$ for the quotient maps. Then we have a commuting square, natural in the space X ,

$$\begin{array}{ccc} X_{A_G} & \xlongequal{\sim} & (K \backslash X)_{A_{G/K}} \\ \downarrow & & \downarrow \\ A_G & \xrightarrow{A_\phi} & A_{G/K}, \end{array}$$

where the vertical maps are the respective structure maps, and the top map is $(K \backslash X)_{A_\phi} \circ p_{A_G}$.

Homogeneous spaces: Let $j: H \hookrightarrow G$ be the inclusion of a closed subgroup. Let $\iota: \text{pt} \rightarrow G/H$ denote the inclusion of the point $1H$. Then we have an isomorphism

$$I_H^G: (G/H)_{A_G} \cong A_H,$$

fitting into the commuting diagram

$$\begin{array}{ccc}
 & (G/H)_{A_H} & \\
 (G/H)_{A_j} \swarrow & & \nwarrow \iota_{A_H} \\
 (G/H)_{A_G} & \xrightarrow[\sim]{I_H^G} & A_H \\
 \pi_{A_G} \searrow & & \swarrow A_j \\
 & A_G &
 \end{array}$$

Künneth: For a G -space X and an H -space Y , we have a commuting square, natural in all ingredients,

$$\begin{array}{ccc}
 (X \times Y)_{A_{G \times H}} & \xrightarrow{\sim} & X_{A_G} \times Y_{A_H} \\
 \downarrow & & \downarrow \\
 A_{G \times H} & \xrightarrow{\sim} & A_G \times A_H,
 \end{array}$$

whenever the corners are defined. In the special case that $G = H = T$ is a compact torus, we have an isomorphism over A_T

$$(X \times Y)_{A_T} \cong X_{A_T} \times_{A_T} Y_{A_T}.$$

Each of these properties can be reformulated in terms of the sheaves $\mathcal{F}_G(X)$, where the obvious generalization for pairs can be formulated (see [GKV95]). The last property that we need is stated most naturally in terms of the reduced theory.

Odd coefficients: Let ϱ be an odd-dimensional orthogonal representation of G . Then $\tilde{\mathcal{F}}_G$ vanishes on the corresponding representation sphere:

$$\tilde{\mathcal{F}}_G(SS^\varrho) = \{0\}.$$

For elliptic cohomology, the change of groups property, the Künneth property and the vanishing of the odd coefficients follow immediately from the construction of $\mathcal{E}ll_G$ and the corresponding properties of H_G .

Proposition 5.1. *Assuming the change of groups and Künneth properties, the homogeneous spaces property and the induction axiom are equivalent.*

PROOF : It is shown in [GKV95, (1.7.5)] that the induction axiom implies the homogeneous spaces property. The other direction is proved by cellular induction: if K acts freely on the orbit G/H then the composite

$$H \hookrightarrow G \rightarrow K \backslash G$$

is still injective, and we have

$$(K \backslash G/H)_{A_\phi} \circ p_{A_G} \circ I_H^G = I_H^{K \backslash G}.$$

Hence $(K \backslash G/H)_{A_\phi} \circ p_{A_G}$ is an isomorphism. \square

We saw in (1) that the homogeneous spaces property holds if G is a compact torus. It follows that the induction axiom holds for the inclusion $H \subseteq T$ of any closed subgroup of a compact torus. In particular, $G_{A_T} = (T \backslash G)_{A_1}$, and hence

$$G_{A_G} = \text{spec}(\mathbb{C}). \quad (7)$$

Further, we saw in Theorem 4.6 that the homogeneous spaces property holds if $H \subseteq G$ are compact and connected of equal rank and $\pi_1(G)$ is torsion free.

Proposition 5.2. *Let G and K be compact connected Lie groups. Then the induction axiom holds for the inclusion*

$$K \triangleleft G \times K.$$

PROOF : Write T and T' for the maximal tori of G and K . Using Künneth, (1) and (7), we see that the homogeneous spaces property holds for any inclusion of the form

$$j \times 1: H \times \{1\} \longrightarrow T_G \times K.$$

The case $G = T_G$ now follows from the proof of Proposition 5.1. In the general case, we have

$$\begin{aligned} X_{A_{G \times K}} &\cong (X_{A_{T \times T'}}) / (W_G \times W_K) \\ &\cong (X_{A_{T \times K}}) / W_G \\ &\cong ((K \backslash X)_{A_T}) / W_G \\ &\cong (K \backslash X)_{A_G}. \end{aligned}$$

This completes the proof. \square

5.2 Classifying maps

Let X be a compact G -manifold, and let

$$\xi: P \longrightarrow X$$

be a G -equivariant principal K -bundle on X . We make the convention that both groups act from the left and that the actions commute.

Write $G \ltimes X$ for the translation groupoid with objects X , arrows

$$x \xrightarrow{g} gx$$

and composition given by composition in G . Similarly, we have the translation groupoids $(G \times K) \ltimes P$ and $K \ltimes \text{pt}$.

Definition 5.3. The *classifying map* of ξ is the generalized map of Lie groupoids

$$f_\xi: G \ltimes X \xleftarrow{\simeq} (G \times K) \ltimes P \longrightarrow K \ltimes \text{pt}.$$

Definition 5.4. The *universal principal K -bundle* is the K -equivariant principal bundle

$$\xi_{\text{univ}}: K \ltimes K \longrightarrow K \ltimes \text{pt},$$

where the two left-actions of K are as follows: as a K -equivariant space, K carries the action of K on itself by left-multiplication. It is a principal K -bundle over the one point space via the action $(k_1, k_2) \mapsto k_2 k_1^{-1}$.

The nomenclature is justified by the following Lemma, which follows directly from the definitions.

Lemma 5.5. *We have an isomorphism*

$$f_\xi^*(\xi_{\text{univ}}) \cong \xi.$$

More can be said here: the assignment

$$\xi \longmapsto f_\xi$$

wants to be an equivalence from the category of principal K -bundles over $G \ltimes X$ to the category of generalized maps (i.e., zig-zags like the one in Definition 5.3), and in fact the former has been used to define the latter, see [Ler10] and [HS87].

Corollary 5.6. *In the situation of Definition 5.3, assume that G is trivial. Then the Borel construction functor, applied to f_ξ , returns the zig-zag*

$$X \xleftarrow{\cong} EK \times_K P \longrightarrow BK.$$

Choosing a homotopy inverse to the first map, we obtain the more familiar classifying map from X to the classifying space of K .

PROOF : This follows, since Borel construction commutes with pull-backs. \square

Example 5.7 (Representations). Let $\varrho: G \rightarrow U(n)$ be a complex representation of G , and let ξ_n be the universal principal $U(n)$ -bundle as in Definition 5.4. Consider the action of G on $U(n)$ by left multiplication with $\varrho(g)$. This makes ξ_n into a G -equivariant principal $U(n)$ -bundle over the one point space. The equivalence

$$G \ltimes \text{pt} \xleftarrow{\cong} (G \times U(n)) \ltimes U(n)$$

of Definition 5.3 has the quasi-inverse

$$g \longmapsto (g, \varrho(g)) \in \text{Stab}(1).$$

Hence the classifying map f_{ξ_n} is equivalent to

$$\varrho: G \ltimes \text{pt} \longrightarrow U(n) \ltimes \text{pt}.$$

Example 5.8 (The splitting principle). Assume that K is a compact connected Lie group and $i: T \hookrightarrow K$ the inclusion of its maximal torus. Then ξ may be factored as the composite

$$\xi: P \xrightarrow{\zeta} T \backslash P \xrightarrow{q} X,$$

where ζ is the quotient map by the T -action. So, ζ is a principal T -bundle, while the fiber of q is the flag variety $T \backslash K$. Over the total space $T \backslash P$ of this flag bundle, the structure group of ξ can be reduced to T . Let

$$\zeta[K]: K \times_T P \longrightarrow T \backslash P$$

be the principal K -bundle obtained from ζ by associating the fiber K . Then

$$q^*(\xi) \cong \zeta[K].$$

This fact is known as the *splitting principle*.

In terms of classifying maps, the splitting principle amounts to the commutativity of the diagram

$$\begin{array}{ccccc} f_\zeta: G \ltimes (T \setminus P) & \xleftarrow{\cong} & (G \times T) \ltimes P & \longrightarrow & T \ltimes \text{pt} \\ \downarrow q & & \downarrow & & \downarrow i \\ f_\xi: G \ltimes X & \xleftarrow{\cong} & (G \times K) \ltimes P & \longrightarrow & K \ltimes \text{pt}. \end{array}$$

Definition 5.9 (Characteristic class). Let $\xi: P \rightarrow X$ be a G -equivariant principal K -bundle with classifying map f_ξ . Then Proposition 5.2 yields a map of schemes

$$c_\xi: X_{A_G} \xlongequal{\quad} P_{G \times K} \longrightarrow A_K.$$

We will refer to c_ξ as the *Ginzburg-Kapranov-Vasserot characteristic class* of ξ .

6 Thom sheaves

Let $\xi: V \rightarrow X$ be a G -equivariant complex vector bundle. We will write X^ξ for the Thom space of ξ and $z: X_+ \hookrightarrow X^\xi$ for the zero section. Applying the reduced theory, we obtain a locally free rank one⁶ module sheaf $\tilde{\mathcal{F}}_G(X^\xi)$ over $\mathcal{F}_G(X)$.

Definition 6.1. The *Thom sheaf* of ξ is the line bundle \mathbb{L}_G^ξ over X_{A_G} characterized (up to isomorphism) by

$$\pi_{A_G*} \left(\mathbb{L}_G^\xi \right)^{-1} \cong \tilde{\mathcal{F}}_G(X^\xi).$$

Note that our convention differs from that in [GKV95, 2.1], where the inverse of \mathbb{L}_G^ξ is referred to as the Thom sheaf. The *Euler map* is the map

$$\eta_G^\xi: \mathcal{O}_{X_{A_G}} \longrightarrow \mathbb{L}_G^\xi$$

induced by the zero section $z: X \rightarrow X^\xi$ (compare [GKV95, (2.6)]). If the group G is understood, we drop it from the notation.

⁶For cohomology and K -theory this is a classical result. In the elliptic case it is an immediate consequence of [Gro07, 2.6] and the W -equivariance of the cohomology Thom isomorphism.

6.1 Properties of the Thom sheaf

The following properties of the Thom sheaf are reformulations of well-known facts about Thom classes in cohomology and K -theory. We deduce the elliptic case, whenever the groups involved have been defined.

Naturality: Let $f: X \rightarrow Y$ be a G -equivariant map, and let ξ be a complex G -vectorbundle over Y . Then we have a commuting diagram of sheaves over X_{A_G}

$$\begin{array}{ccc} & \mathcal{O}_{X_{A_G}} & \\ f_{A_G}^* \eta^\xi \swarrow & & \searrow \eta^{f^* \xi} \\ \mathbb{L}_G^{f^* \xi} & \xlongequal{\sim} & f_{A_G}^* \mathbb{L}_G^\xi. \end{array}$$

PROOF : The map

$$\tilde{\mathcal{F}}_G(f^\xi): \tilde{\mathcal{F}}_G(Y^\xi) \longrightarrow \tilde{\mathcal{F}}_G(X^{f^* \xi})$$

is a map of $\mathcal{F}_G(Y)$ -module sheaves. Hence it corresponds to a map

$$(\mathbb{L}^\xi)^{-1} \longrightarrow f_{A_G*}(\mathbb{L}^{f^* \xi})^{-1},$$

whose adjoint

$$f_{A_G}^*(\mathbb{L}^\xi)^{-1} \longrightarrow (\mathbb{L}^{f^* \xi})^{-1}$$

is an isomorphism. In the elliptic case, the last statement follows from [Gro07, 2.6] and the W -equivariance of the Thom isomorphism in cohomology. \square

Change of groups: Let $\phi: H \rightarrow G$ be a map of Lie groups, and let $\xi: V \rightarrow X$ be a G -equivariant complex vector bundle. Then we have a commuting diagram of sheaves over X_{A_H}

$$\begin{array}{ccc} & \mathcal{O}_{X_{A_H}} & \\ \eta_H^\xi \swarrow & & \searrow A_\phi^* \eta_G^\xi \\ \mathbb{L}_H^\xi & \xlongequal{\sim} & X_{A_\phi}^* \mathbb{L}_G^\xi. \end{array}$$

PROOF : Consider the map of locally free rank one $\mathcal{F}_H(X)$ module sheaves

$$\mathcal{F}_H(X) \otimes_{A_\phi^* \mathcal{F}_G(X)} A_\phi^* \tilde{\mathcal{F}}_G(X^\xi) \longrightarrow \tilde{\mathcal{F}}_H(X^\xi).$$

For $[a] \in A_H$, we have an equality of fixed point sets

$$X^{T(a)} = X^{T(\phi(a))}.$$

Hence [Gro07, 2.6] implies that the above map is an isomorphism at the stalk $[a]$. \square

Induction: Assume that the induction axiom holds for the inclusion of a normal subgroup $K \triangleleft G$ and that X is a G -complex on which the action of K is free. Then the change of groups isomorphism $X_{A_G} \cong (K \backslash X)_{A_K}$ is covered by an isomorphism of line bundles identifying \mathbb{L}_G^ξ with $\mathbb{L}_{G/K}^{K \backslash \xi}$ and η_G^ξ with $\eta_{G/K}^{K \backslash \xi}$.

Multiplicativity: Given equivariant complex vector bundles ξ over a G -space X and ζ over an H -space Y , the Künneth isomorphism $(X \times Y)_{A_{G \times H}} \cong X_{A_G} \times Y_{A_H}$ is covered by an isomorphism of line bundles identifying $\mathbb{L}_{G \times H}^{\xi \oplus \zeta}$ with the external tensor product $\mathbb{L}_G^\xi \otimes \mathbb{L}_H^\zeta$ and $\eta_{G \times H}^{\xi \oplus \zeta}$ with $\eta_G^\xi \otimes \eta_H^\zeta$. In the special case where $G = H$, we get a commuting square of sheaves over A_G

$$\begin{array}{ccc} & \mathcal{O}_{X_{A_G}} & \\ \swarrow & & \searrow \\ \mathbb{L}_G^{\xi \oplus \zeta} & \xlongequal{\sim} & \mathbb{L}_G^\xi \otimes_{\mathcal{O}_{A_G}} \mathbb{L}_G^\zeta. \end{array}$$

PROOF : The induction and (external) multiplicativity properties follow directly from the induction axiom and the Künneth property of \mathcal{F}_G in their formulation for pairs. The internal Künneth for $\mathbb{L}_T^{\xi \oplus \zeta}$ follows from the external Künneth and the change of groups property. \square

Universal bundles: The Thom sheaf of the universal complex line bundle ξ_1 over pt (compare Definition 5.4) is the line bundle

$$\mathbb{L}_{U(1)}^{\xi_1} \cong \mathcal{L}(0)$$

of the divisor (0) on $A_{U(1)}$. Recall that $A_{U(1)}$ equals \mathbb{C} or \mathbb{C}^\times , in which case we replace 0 by 1, or $\mathbb{C}/\langle \tau, 1 \rangle$. The Euler map of ξ_1 is the canonical inclusion

$$\mathcal{O}_{A_{U(1)}} \longrightarrow \mathcal{L}(0).$$

Consider the universal complex n -vector bundle ξ_n over pt. Let T be the maximal torus of $U(n)$. Then

$$\mathbb{L}_T^{\xi_n} \cong \bigotimes_{i=1}^n p_i^* \mathcal{L}(0)$$

where

$$p_i: A_{U(1)}^n \longrightarrow A_{U(1)}$$

is the projection to the i th factor. This is the line bundle associated to the divisor

$$\sum_{t=1}^n \ker(p_t),$$

and the Euler map $\eta_T(\xi)$ is its canonical inclusion of $\mathcal{O}_{A_{U(1)}^n}$ inside it. The $U(n)$ -equivariant Thom sheaf and Euler map are obtained by taking the S_n -invariant parts of \mathbb{L}_T and η_T . We will write η_n for the n th universal Euler map.

The following result, which determines all Thom sheaves up to isomorphism, follows immediately from the list of properties above.

Theorem 6.2. *Let $\xi: V \rightarrow X$ be a G -equivariant vector bundle, and let c_ξ be its Ginzburg-Kapranov-Vasserot characteristic class (c.f. Definition 5.9). Then we have a commuting square*

$$\begin{array}{ccc} \mathcal{O}_{X_{A_G}} & \xlongequal{\quad} & c_\xi^* \mathcal{O}_{A_{U(1)}^n/S_n} \\ \eta(\xi) \downarrow & & \downarrow c_\xi^*(\eta_n) \\ \mathbb{L}_G^\xi & \xlongequal[\sim]{} & c_\xi^* \mathbb{L}^{\xi_n}. \end{array}$$

Example 6.3. Let $\lambda \neq 0$ be a weight of T , and let $j: K_\lambda \hookrightarrow T$ be the kernel of $e^{2\pi i \lambda}$. Consider the T -equivariant line bundle $\xi_\lambda: \mathbb{C}_\lambda \rightarrow \text{pt.}$ By Example 5.7, we have a short exact sequence

$$A_{K_\lambda} \xrightarrow{A_j} A_T \xrightarrow{c_{\xi_\lambda}} A.$$

Hence the Thom sheaf of ξ_λ is

$$\mathbb{L}^{\xi_\lambda} \cong \mathcal{L}(A_{K_\lambda}),$$

the line bundle on A_T associated to the divisor A_j .

Example 6.4. Let $\varrho: G \rightarrow U(n)$ be a complex representation, viewed as a G -equivariant complex vector bundle over the one point space. By Example 5.7, we have an isomorphism

$$\mathbb{L}_G^\varrho \cong A_\varrho^* \mathbb{L}_{U(n)}^{\xi_n}$$

of sheaves over A_G .

Example 6.5. If $\varrho: G \rightarrow U(1)$ is the one dimensional trivial representation, then A_ϱ factors through the inclusion of zero $i_0: A_1 \rightarrow A_{U(1)}$. Hence

$$\tilde{F}_1(SS^2) \cong i_0^* \mathcal{I}(0)$$

is identified with the sheaf of invariant differentials on $A_{U(1)}$,

$$\omega := \mathcal{I}(0)/\mathcal{I}(0)^2|_0.$$

Writing ω also for its pull-back to A_G , we obtain

$$\tilde{F}_G(SS^{2n}) \cong \omega^{\otimes n}.$$

This shows that the Periodicity Axiom (1.5.5) in [GKV95] is, in part, redundant. We will come back to this in Section 6.2.

Example 6.6. Let

$$\chi_\varrho = e^{2\pi i \lambda_1} + \dots + e^{2\pi i \lambda_n}$$

be the character of ϱ , with $\lambda_k \in \Lambda \setminus \{0\}$ for all k . By Example 6.3, we may identify the G -equivariant Thom sheaf \mathbb{L}_G^ϱ over A_T/W with the sheaf of S_n -invariant sections of

$$\bigotimes_{i=0}^n \mathcal{L}(A_{K_{\lambda_i}}).$$

Corollary 6.7. *Let $\varrho: T \rightarrow U(n)$ be a complex representation of T with character*

$$\chi_\varrho = e^{2\pi i \lambda_1} + \dots + e^{2\pi i \lambda_n},$$

$\lambda_i \neq 0$, and write S_ϱ^{2n-1} for its unit sphere inside \mathbb{C}^n . Then we have an isomorphism

$$(SS_\varrho^{2n-1})_{A_T} \cong \bigcap_{i=1}^n A_{K_{\lambda_i}},$$

where the right-hand side stands for the scheme theoretic intersection over A_T .

PROOF : We have a cofiber sequence

$$(SS_{\varrho}^{2n-1})_+ \longrightarrow SS^0 \longrightarrow SS^{\varrho},$$

whose second map is the zero section z in the definition of η_T^{ϱ} . Applying \widetilde{F}_T , we obtain the short exact sequence

$$\bigotimes_{i=1}^n \mathcal{I}(A_{K_{\lambda_i}}) \longrightarrow \mathcal{O}_{A_T} \longrightarrow \mathcal{F}_T(SS_{\varrho}^{2n-1}),$$

where the first map is the canonical inclusion. □

Example 6.8. Consider the representation

$$k\xi_1 := \xi_1 \oplus \cdots \oplus \xi_1$$

of $U(1)$. Then

$$(SS_{k\xi_1}^{2k-1})_{A_{U(1)}} = (0)^{[k]}$$

is the k th infinitesimal neighbourhood of 0 inside $A_{U(1)}$.

We are now in a position to give the promised proof of the Completion Theorem from the formal properties of X_{A_T} .

PROOF OF THEOREM 3.1: We follow the outline in [GKV95, (1.7.2)]. Writing T as the product of r copies of $U(1)$, we may build ET from the equivariant skeleta

$$ET^{(2k-1)} := SS_{k\xi_1}^{2k-1} \times \cdots \times SS_{k\xi_1}^{2k-1},$$

where the notation is as in the last example. The induction axiom gives a commuting diagram

$$\begin{array}{ccc} (ET^{(2k-1)} \times_T X)_{A_1} & \xlongequal{\sim} & (ET^{(2k-1)} \times X)_{A_T} \\ \downarrow & & \downarrow \\ A_1 & \xleftarrow{\quad} & A_T. \end{array}$$

A combination of the internal and external Künneth properties, together with the last example, yields an isomorphism of schemes over A_T

$$(ET^{(2k-1)} \times X)_{A_T} \cong \left(\prod_{j=1}^r (0)^{[k]} \right) \times_{A_T} X_{A_T}$$

Letting k vary, the right-hand side becomes an ind-scheme over A_1 , isomorphic to the formal completion $(X_{A_T})_{\widehat{0}}$. \square

6.2 $RO(G)$ -grading and periodicity

We are now ready to define the full theory $\mathcal{E}ll_G^*(-)$, graded by the set of orthogonal representations contained in an indexing universe (see [May96, p.154]) and their formal differences. For any such universe, there is a cofinal system of representations of the form

$$\varrho: G \longrightarrow U(n) \longrightarrow O(2n).$$

Hence it suffices to define the groups

$$\widetilde{\mathcal{E}ll}_G^{\varrho-\sigma}(X) := \mathbb{L}^{\varrho} \otimes_{\mathcal{O}_{\mathcal{M}_G}} \widetilde{\mathcal{E}ll}_G^0(\mathbb{S}^{\sigma} \wedge X),$$

where ϱ is as above and σ is in our universe. The resulting theory satisfies the axioms of a sheaf-valued $RO(G)$ -graded cohomology theory: $\mathcal{E}ll_G$ is a contravariant functor of X and σ and a covariant functor of ϱ . Each $\mathcal{E}ll_G^{\varrho-\sigma}(-)$ is exact on cofibre sequences and sends wedges to products. There are suspension isomorphisms

$$s^{\varrho}: \widetilde{\mathcal{E}ll}_G^{*+\varrho}(\mathbb{S}^{\varrho} \wedge X) \xrightarrow{\cong} \widetilde{\mathcal{E}ll}_G^*(X),$$

natural in X and the orthogonal representation ϱ , and satisfying

$$s^{\varrho \oplus \sigma} = s^{\sigma} \circ s^{\varrho}.$$

Remark 6.9. These axioms are immediate from the definitions. Note that exactness is checked on stalks, and we cannot expect sections over an open $\Gamma(U, \mathcal{E}ll_G^*(-))$ to be exact. In other words, a sheaf-valued cohomology theory is *not* the same thing as a sheaf of cohomology theories.

The theory of Thom sheaves is extended to virtual equivariant complex vector bundles, and we set

$$\mathcal{E}l_G^{*+\xi}(X) := \widetilde{\mathcal{E}l}_G^*(X^{-\xi}).$$

Finally, we have periodicity isomorphisms

$$\begin{aligned} \widetilde{\mathcal{E}l}_G^*(\mathbb{S}^\varrho \wedge X) &\cong \mathbb{L}^{-\varrho} \otimes_{\mathcal{E}l_G} \widetilde{\mathcal{E}l}_G^*(X) \\ &\cong \widetilde{\mathcal{E}l}_G^{*- \varrho} \otimes_{\mathcal{E}l_G^*} \widetilde{\mathcal{E}l}_G^*(X) \end{aligned}$$

for complex representations ϱ .

7 Euler classes

Since the elliptic Thom sheaves are in general non-trivial, the notion of Euler class does not have an immediate generalization to elliptic cohomology. The different authors make different choices on this matter, see [Ro01, p.10], [And03, 5.3] and [GKV95, (2.6)].

7.1 Thom isomorphisms

Let ξ_1 be the universal complex line bundle of Definition 5.4, and recall that its Thom sheaf is the invertible sheaf

$$\mathbb{L}_{U(1)}^{\xi_1} \cong \mathcal{L}(0)$$

over $A_{U(1)}$. For the additive or multiplicative group this divisor is principal:

$$\begin{aligned} (0) &= \operatorname{div}(x) && \text{on } \mathbb{C} \text{ and} \\ (1) &= \operatorname{div}(1 - z) && \text{on } \mathbb{C}^\times. \end{aligned}$$

The *universal Euler classes* in cohomology and K -theory are the functions

$$e(\xi_1) = \begin{cases} x & \text{on } \mathbb{C} \text{ and} \\ (1 - z) & \text{on } \mathbb{C}^\times, \end{cases}$$

and the *universal Thom isomorphisms* are

$$\begin{aligned} \vartheta: \mathcal{O}_{A_{U(1)}} &\xrightarrow{\cong} \mathcal{L}(0) \\ f &\longmapsto \frac{f}{e(\xi_1)} \end{aligned}$$

(replace (0) by (1) for K -theory). As a consequence, all Thom sheaves in these theories are trivialized, and the theories possess Chern classes for complex vector bundles.

We will be particularly interested in the case where the complex vector bundle $\xi: V \rightarrow X$ comes equipped with a spin structure on the underlying real bundle. In this case, the Ginzburg-Kapranov-Vasserot characteristic class factors as

$$\begin{array}{ccc} & A_{U^2(n)} & \\ \tilde{c}_\xi \nearrow & & \searrow v \\ X_{A_G} & \xrightarrow{c_\xi} & A_{U(n)}, \end{array}$$

where $A_{U^2(n)}$ is the pull-back in the cartesian square

$$\begin{array}{ccc} A_{U^2(n)} & \xrightarrow{u} & A_{U(1)} \\ v \downarrow & & \downarrow \cdot 2 \\ A_{U(n)} & \xrightarrow{A_{\det}} & A_{U(1)}. \end{array}$$

Example 7.1. In the multiplicative case, a point in $A_{U^2(n)}$ consists of

$$(z_1, \dots, z_n) \in (\mathbb{C}^\times)^n / S_n$$

together with a choice of square root

$$(z_1 \cdots z_n)^{\frac{1}{2}}.$$

Definition 7.2. We write

$$\mathbb{L}_{U^2(n)}^{\xi_n} := v^* \mathbb{L}_{U(n)}^{\xi_n}.$$

for the universal Thom sheaf for such n -dimensional complex bundles with spin structure.

In K -theory $\mathbb{L}_{U^2(n)}^{\xi_n}$ is the target of the *Atiyah-Bott-Shapiro Thom isomorphism*

$$\begin{aligned} \vartheta_{ABS}: \mathcal{O}_{A_{U^2(n)}} &\xrightarrow{\cong} \mathbb{L}_{U^2(n)}^{\xi_n} \\ f &\longmapsto \frac{f}{e'(\xi_n)}, \end{aligned}$$

with

$$e'(\xi_n) = \left(z_1^{\frac{1}{2}} - z_1^{-\frac{1}{2}} \right) \cdots \left(z_n^{\frac{1}{2}} - z_n^{-\frac{1}{2}} \right).$$

7.2 Theta functions and elliptic Euler classes

On the elliptic curve $E = \mathbb{C}/2\pi i\langle \tau, 1 \rangle$ the divisor (0) is no longer principal. In this case, the Thom isomorphisms above are replaced by the theta function formalism: let ϱ be a complex representation of G with character

$$e^{2\pi i\lambda_1} + \dots + e^{2\pi i\lambda_n},$$

$\lambda_k \in \Lambda \setminus \{0\}$. Then the first Pontrjagin class

$$p_1(\varrho) = \sum_{k=1}^n \lambda_k \otimes \lambda_k$$

is an integer-valued positive definite symmetric bilinear form on Λ^\vee . If ϱ admits a spin structure then we have

$$p_1(\varrho)(x, x) \in 2\mathbb{Z} \tag{8}$$

for all $x \in \Lambda^\vee$.

Definition 7.3. Let I be a positive definite symmetric bilinear form on Λ^\vee , and assume that I satisfies (8). Then the *Looijenga line bundle* associated to I is the invertible sheaf \mathcal{L}_I on \mathcal{M}_T^h with sections

$$\mathcal{L}_I(U) = \{f \in \Gamma \mathcal{O}_{y^{-1}U}^h \mid f(q^x z) = q^{-\frac{1}{2}I(x,x)} z^{-I^\sharp(x)} f(z)\}.$$

Here $x \in \Lambda^\vee$, and q^x stands for the image of τx under $\exp: \mathfrak{t}_\mathbb{C} \rightarrow T_\mathbb{C}$, while y is the quotient map $T_\mathbb{C} \rightarrow \mathcal{M}_T$. So, if $T = U(1)$ then $q^x = e^{2\pi i \tau x}$. We write

$$\Theta_\varrho = \Gamma \mathcal{L}_{p_1(\varrho)}$$

for the global sections of $\mathcal{L}_{p_1(\varrho)}$ and refer to elements of Θ_ϱ as *Looijenga theta functions (of level $p_1(\varrho)$)*.

Definition 7.4. Let $\varrho: G \rightarrow U(n)$ be as above. We are still assuming that we have a spin structure on ϱ . The “*elliptic Euler class*” of ϱ is the function on $T_\mathbb{C}$ defined by

$$e_{\text{ell}}(\varrho) := (-1)^n \prod_{i=1}^n \sigma(q, z^{\lambda_i}),$$

where

$$\sigma(q, z) = (z^{\frac{1}{2}} - z^{-\frac{1}{2}}) \prod_{n \geq 1} \frac{(1 - q^n z)(1 - q^n z^{-1})}{(1 - q^n)^2}$$

is the Weierstrass sigma function, and for $z = \exp(x) \in T_{\mathbb{C}}$ and $\lambda \in \Lambda$ we are using the notation

$$z^\lambda := e^{2\pi i \lambda(x)}.$$

In elliptic cohomology, the role of the Thom isomorphism is replaced by the isomorphism of line bundles over \mathcal{M}_G

$$\begin{aligned} \vartheta: \mathcal{L}_{p_1(\varrho)}^W &\xrightarrow{\cong} \mathbb{L}_G^{\varrho} \\ f &\longmapsto \frac{f}{e_{ell}(\varrho)}. \end{aligned}$$

Presumably, these notions generalize to yield a theta function description for the elliptic Thom sheaf of any equivariant $U^2(n)$ -bundle over a nice enough base (for instance, an equivariantly formal space). We do not pursue this here.

Remark 7.5. Ando has considered equivariant elliptic cohomology with twisted coefficients, where the twist comes from an element $\beta \in H^4(BG; \mathbb{Z})$. He does so by extending Looijenga's definition of \mathcal{L}_β to β not satisfying (8). Let G be connected, and let ϱ be an even dimensional orthogonal representation of G . Then a similar argument to the one above yields an isomorphism between $\mathcal{E}ll_G^{\varrho}(\text{pt})$ and $\mathcal{L}_{p_1(\varrho)}^W$. So, the $RO(G)$ -graded coefficients are contained in Ando's picture.

7.3 Push-forwards

Let X and Y be compact, closed smooth manifolds, and let $f: X \rightarrow Y$ be a complex oriented map in the sense of [Qui71]. That means that we have a factorization

$$\begin{array}{ccc} & & E \\ & \nearrow i & \downarrow \xi \\ X & \xrightarrow{f} & Y, \end{array}$$

where ξ is a complex vector bundle and the normal bundle ν of i is equipped with a complex structure.

Definition 7.6. For such a complex oriented map f , one defines the *relative Thom sheaf* as

$$\mathbb{L}(f) = f_{A_G*} \mathbb{L}^{-\nu} \otimes \mathbb{L}^{\xi}.$$

This is a sheaf over Y_{A_G} . The *push-forward* along f is the map

$$f_! : \mathbb{L}(f) \longrightarrow \mathcal{O}_{Y_{A_G}}$$

of sheaves over Y_{A_G} that is adjoint to the map

$$f_{A_G*} \mathbb{L}^{-\nu} \longrightarrow \mathbb{L}^{-\xi}$$

induced by the Pontrjagin-Thom collapse.

The following lemma is immediate from the definitions.

Lemma 7.7 (Localization Lemma). *Let X be as above with a smooth T -action. Let $i : X^T \hookrightarrow X$ be the inclusion of the fixed points and assume that we are given a T -equivariant complex structure on the normal bundle ν of i . Then we have a commuting diagram*

$$\begin{array}{ccc} & \mathcal{F}_T^*(X) & \\ i^* \swarrow & & \nwarrow i_! \\ \mathcal{F}_T^*(X^T) & \xleftarrow{z^*} & \mathcal{F}_T^{*- \nu}(X^T), \end{array}$$

where z is the zero section of $(X^T)^\nu$.

Note that the trivial representation does not turn up as a summand inside ν .

Corollary 7.8. *In the situation of the Localization Theorem 4.1, assume that the normal bundle is equipped with a T -equivariant complex structure. Let $\Delta(\nu) \subseteq A_T$ be the closed subset*

$$\Delta(\nu) = \bigcup_{\mathbb{C}_\lambda \subseteq \nu} \ker(A_{e^\lambda}).$$

Then, restricted to $A_T \setminus \Delta(\nu)$, the map i^ becomes an isomorphism with inverse $i_! \circ (z^*)^{-1}$.*

8 Character Formulas

8.1 Induced representations

Let G be a compact connected Lie group with maximal torus T , and let $B \subseteq G_{\mathbb{C}}$ be a Borel subgroup of its complexification. Such a choice of B is equivalent to a choice of positive roots of G . It endows the flag variety

$$G/T \cong G_{\mathbb{C}}/B$$

with a complex structure such that the tangent space at the coset of 1 is the complex T -representation

$$\mathfrak{g}/\mathfrak{t} \cong_{\mathbb{C}} \bigoplus_{\alpha \in \mathcal{R}_-} \mathbb{C}_{\alpha}.$$

Similarly, if $H \subseteq G$ is a connected subgroup containing T and P_H the parabolic subgroup corresponding to H , we have a complex structure on the homogenous space

$$G/H \cong G_{\mathbb{C}}/P_H.$$

Definition 8.1. In this situation, we define the map

$$\text{ind}: R(H) \longrightarrow R(G)$$

as the composite

$$\begin{array}{ccc} K_H & \xlongequal{\sim} & K_G(G/H) \\ \vartheta \parallel & & \parallel \vartheta \\ K_H^{\mathfrak{g}/\mathfrak{h}} & \xlongequal{\sim} & K_G^{\tau}(G/H) \xrightarrow{\pi_!} K_G. \end{array}$$

Here, and I apologize for this notation, \mathfrak{h} is the Lie algebra of H , *not* a Cartan subalgebra. Further

$$\tau \cong G \times_H \mathfrak{g}/\mathfrak{h}$$

is the tangent bundle of G/H , and π is the unique map from G/H to the one point space. The push-forward $\pi_!$ is as defined in Section 7.3.⁷

⁷A more common definition of the push-forward $\pi_!$ in K -theory or cohomology is the composite of our $\pi_!$ with ϑ . The reason for our convention is that it generalizes to elliptic cohomology in a canonical way.

The Atiyah-Singer index theorem identifies our definition of ind with the definition of induction found in the representation-theory literature:

Theorem 8.2 (c.f. [AS68] or [HBJ92, 5.4]). *Let $\varrho: H \rightarrow GL(V)$ be a complex representation. Then*

$$\text{ind}([\varrho]) = \sum (-1)^i H^i(G/H, \mathcal{O}(G \times_H V))$$

is the induced representation of ϱ . Here $\mathcal{O}(G \times_H V)$ is the sheaf of holomorphic sections of $G \times_H V$.

8.2 The Weyl character formula

We will now compute the character of these induced representations. As in Theorem 4.6, we let W_H and W_G be the respective Weyl groups, and we let

$$i: F \longrightarrow G/H$$

be the inclusion of the T -fixed points $F := (G/H)^T$. Recall that F can be identified with the set W_G/W_H , and that we have

$$i^* \tau \cong_T \coprod_{[w] \in F} (\mathfrak{g}/\mathfrak{h})^w$$

(conjugation by w on the right-hand side). We have a commuting diagram

$$\begin{array}{ccccc}
K_H \simeq K_G(G/H) & \xlongequal{\vartheta} & K_G^T(G/H) & \xrightarrow{\pi!} & K_G \\
\downarrow \text{res} & & \downarrow \text{res} & & \downarrow \text{char} \\
K_T(G/H) & \xlongequal{\vartheta} & K_T^T(G/H) & \xrightarrow{\pi!} & K_T \\
\downarrow i^* & & \downarrow i^* & \nearrow i! & \uparrow \sum_{[w] \in F} (-)_w \\
\bigoplus_{[w] \in F} K_T & \xlongequal{\vartheta} & \bigoplus_{[w] \in F} K_T^{(\mathfrak{g}/\mathfrak{h})^w} & \xleftarrow{z^*} & \bigoplus_{[w] \in F} K_T
\end{array} \tag{9}$$

where

$$z: F \longrightarrow F^{i^*\tau}$$

is the zero section.

The top row of (9) is the map ind of definition 8.1. The composite of the vertical arrows on the left sends an H -representation ϱ to $(\chi_\varrho^w)_{[w] \in F}$ (the character of ϱ and its conjugates under W_G). The composite at the bottom is multiplication by the Euler class of $i^*\tau$. On the $[w]$ th summand, this is

$$e(i^*\tau)_{[w]} = \prod_{\alpha \in \mathcal{R}} (1 - e^{w(\alpha)}).$$

Here

$$\mathcal{R} := \mathcal{R}_-^G \setminus \mathcal{R}_-^H$$

consists of the negative roots of G that are not roots of H . Using the Localization Lemma (see Corollary 7.8), we can deduce Weyl's character formula:

Theorem 8.3 (Weyl). *Let ϱ be a representation of H . Then the character of its induced representation equals*

$$\chi_{\text{ind}(\varrho)} = \sum_{[w] \in F} \frac{\chi_\varrho^w}{\prod_{\alpha \in \mathcal{R}} (1 - e^{2\pi i w(\alpha)})}. \quad (10)$$

To be precise, the Localization Lemma implies the equality (10) in the localized ring

$$R(T)[e(\mathfrak{g}/\mathfrak{h})^{-1}].$$

Since $R(T)$ maps injectively into this localization, and $\chi_{\text{ind}(\varrho)}$ is an element of $R(T)$, it makes sense to interpret (10) as a formula in $R(T)$.

Replacing K -theory by cohomology, we obtain a formula for the composite

$$(Bj)^* \circ (Bk)_!,$$

where j and k are the respective inclusions of T and H in G . Namely, it sends a regular function f on A_H to

$$Bj^*(Bk_!(f)) = \frac{\sum_{[w] \in f} \det(w) f^w}{\prod_{\alpha \in \mathcal{R}} \alpha_{\mathbb{C}}}.$$

8.3 The Kac character formula

We now turn our attention to the elliptic case, making the additional assumption that the partial flag variety G/H carries a U^2 -structure. For simplicity of notation, we write

$$Ell_G^*(X) := \Gamma \mathcal{E}ll_G^*(X)^h$$

for the analytic global sections, noting that the statement holds on the level of sheaves with all sheaves pushed forward to \mathcal{M}_G . Let ϱ be a G -representation. The diagram (9) is replaced by

$$(10) \quad \begin{array}{ccccc} \Theta_{\varrho+\mathfrak{g}/\mathfrak{h}}^{W_H} & \xlongequal{\sim} & Ell_G^{\varrho+\tau}(G/H) & \xrightarrow{\pi_!} & Ell_G^{\varrho} \xlongequal{\vartheta} \Theta_{\rho}^{W_G} \\ & & \downarrow \text{res} & & \downarrow \text{char} \\ & & Ell_T^{\varrho+\tau}(G/H) & \xrightarrow{\pi_!} & Ell_T^{\varrho} \xlongequal{\vartheta} \Theta_{\rho} \\ & & \downarrow i^* & \nearrow i_! & \uparrow \sum_{[w] \in F} (-)_w \\ \bigoplus_{[w] \in F} \Theta_{\varrho+(\mathfrak{g}/\mathfrak{h})^w} & \xlongequal{\vartheta} & \bigoplus_{[w] \in F} Ell_T^{\varrho+(\mathfrak{g}/\mathfrak{h})^w} & \xleftarrow{z^*} & \bigoplus_{[w] \in F} Ell_T^{\varrho}. \end{array}$$

As before, we will write *ind* for the composite of the arrows in the top row. The Thom sheaf $\mathbb{L}_T^{\mathfrak{g}/\mathfrak{h}}$ is the line bundle over \mathcal{M}_T associated to the divisor

$$\Delta = \sum_{\alpha \in \mathcal{R}} (A_{K_{\alpha}}),$$

on \mathcal{M}_T . Here $K_{\alpha} = \ker(e^{2\pi i \alpha})$. The T -equivariant Euler class of $\mathfrak{g}/\mathfrak{h}$ is the theta-function

$$e_{ell}(\mathfrak{g}/\mathfrak{h}) = \pm \prod_{\alpha \in \mathcal{R}} (z^{\frac{\alpha}{2}} - z^{-\frac{\alpha}{2}}) \prod_{n \geq 1} \frac{(1 - q^n z^{\alpha})(1 - q^n z^{-\alpha})}{(1 - q^n)^2},$$

where the sign equals $(-1)^{|\mathcal{R}|}$. Set

$$\Phi = \Phi(q) := \prod_{n \geq 1} (1 - q^n)^2.$$

Theorem 8.4. *Let f be an element of $\Theta_{\varrho+\mathfrak{g}/\mathfrak{h}}^{W_H}$. Then we have*

$$ind(f) = (-\Phi)^d \frac{\sum_{[w] \in F} \det(w) w(f)}{\prod_{\alpha \in \mathcal{R}} \left(z^{\frac{\alpha}{2}} - z^{-\frac{\alpha}{2}}\right) \prod_{n \geq 1} (1 - q^n z^\alpha) (1 - q^n z^{-\alpha})}.$$

Here d is the complex dimension of the partial flag variety G/H . Consider now the special case where G is simple and simply connected, and $H = T$ is the maximal torus. Then there is a smallest positive definite bilinear form I_{Lo} satisfying (8). This is the bilinear form considered in [Loo77], and we write \mathcal{L}_{Lo} for the corresponding Looijenga line bundle.⁸ By [Loo77, (3.4)], we have

$$p_1(\mathfrak{g}/\mathfrak{t}) = g \cdot I_{Lo}$$

and hence

$$\mathcal{L}_{p_1(\mathfrak{g}/\mathfrak{t})} = \mathcal{L}_{Lo}^g.$$

Here g is the dual Coxeter number. Assume that we have $p_1(\varrho) = k I_{Lo}$ with $k \in \mathbb{Z}$. Then the top row of (10) becomes a map

$$ind: \Theta_{k+g} \longrightarrow \Theta_k^{W_G},$$

where Θ_k are the Looijenga theta functions of level k .

Definition 8.5 (Looijenga basis). Let $k \in \mathbb{N}$, and let $\lambda \in \Lambda$. The element $\theta_{k,\lambda} \in \Theta_k$ is defined by

$$\theta_{k,\lambda} := \sum_{x \in \Lambda^\vee} q^{(k\phi+\lambda)(x)} e^{2\pi i(kI^\sharp(x)+\lambda)}.$$

Here

$$\phi(x) := \frac{1}{2} I_{Lo}(x, x)$$

and $I^\sharp: \Lambda^\vee \rightarrow \widehat{T}$ is the adjoint of I_{Lo} .

⁸In the notation of [Loo77] this is the line bundle \mathcal{L}^{-1} .

As λ varies over a set of representatives for $\Lambda/kI^\sharp(\Lambda^\vee)$, the $\theta_{k,\lambda}$ form a basis for Θ_k .

Corollary 8.6 (Kac character formula). *In the situation of Theorem 8.4, assume that G is simple and simply connected and let $H = T$ be the maximal torus. Then we have*

$$\text{ind}(\theta_{k+g,\lambda}) = \frac{(-\Phi)^d \cdot \sum_{w \in W_G} \det(w) \cdot \theta_{k+g,w(\lambda)}}{\prod_{\alpha \in \mathcal{R}_-} (e^{\pi i \alpha} - e^{-\pi i \alpha}) \prod_{n \geq 1} (1 - q^n e^{2\pi i \alpha}) (1 - q^n e^{-2\pi i \alpha})}.$$

Up to the factor $\pm \Phi(q)^{d+r}$, which is constant in z , this agrees with the Kac character formula for the positive energy representation of the loop group \mathcal{LG} of level k and heighest weight

$$\lambda + \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} \alpha.$$

For a presentation of the Kac character formula in this form see [PS86, (14.3.4)] or [And00, 11.4].

References

- [AB68] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. II. Applications. *Ann. of Math. (2)*, 88:451–491, 1968.
- [AB84] M. F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.
- [AC83] E. Akyıldız and J. B. Carrell. Zeros of holomorphic vector fields and the Gysin homomorphism. In *Singularities, Part 1 (Arcata, Calif., 1981)*, volume 40 of *Proc. Sympos. Pure Math.*, pages 47–54. Amer. Math. Soc., Providence, RI, 1983.
- [And00] Matthew Ando. Power operations in elliptic cohomology and representations of loop groups. *Trans. Amer. Math. Soc.*, 352(12):5619–5666, 2000.

- [And03] Matthew Ando. The sigma orientation for analytic circle-equivariant elliptic cohomology. *Geom. Topol.*, 7:91–153 (electronic), 2003.
- [AS68] M. F. Atiyah and I. M. Singer. The index of elliptic operators. I. *Ann. of Math. (2)*, 87:484–530, 1968.
- [AS69] M. F. Atiyah and G. B. Segal. Equivariant K -theory and completion. *J. Differential Geometry*, 3:1–18, 1969.
- [BE90] Paul Bressler and Sam Evens. The Schubert calculus, braid relations, and generalized cohomology. *Trans. Amer. Math. Soc.*, 317(2):799–811, 1990.
- [BE92] Paul Bressler and Sam Evens. Schubert calculus in complex cobordism. *Trans. Amer. Math. Soc.*, 331(2):799–813, 1992.
- [BtD85] Theodor Bröcker and Tammo tom Dieck. *Representations of Compact Lie groups*. Graduate Texts in Mathematics, No. 98. Springer-Verlag, 1985.
- [CPZ09] Baptiste Calmès, Victor Petrov, and Kirill Zainoulline. Invariants, torsion indices and oriented cohomology of complete flags. <http://front.math.ucdavis.edu/0905.1341>, May 2009.
- [Gep05] David Gepner. *Homotopy topoi and equivariant elliptic cohomology*. PhD thesis, University of Illinois at Urbana-Champaign, 2005.
- [GKM98] Mark Goresky, Robert Kottwitz, and Robert MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131(1):25–83, 1998.
- [GKV95] Victor Ginzburg, Mikhail Kapranov, and Eric Vasserot. Elliptic algebras and equivariant elliptic cohomology I. (technical report) <http://front.math.ucdavis.edu/9505.5152>, May 1995.
- [GR] Nora Ganter and Arun Ram. Elliptic Schubert calculus. In preparation.
- [Gro07] I. Grojnowski. Delocalised equivariant elliptic cohomology. In *Elliptic cohomology*, volume 342 of *London Math. Soc. Lecture Note Ser.*, pages 114–121. Cambridge Univ. Press, Cambridge, 2007.

- [HBJ92] Friedrich Hirzebruch, Thomas Berger, and Rainer Jung. *Manifolds and modular forms*. Aspects of Mathematics, E20. Friedr. Vieweg & Sohn, Braunschweig, 1992. With appendices by Nils-Peter Skoruppa and by Paul Baum.
- [HK11] Jens Hornbostel and Valentina Kiritchenko. Schubert calculus for algebraic cobordism. *J. Reine Angew. Math.*, 656:59–85, 2011.
- [HS87] Michel Hilsun and Georges Skandalis. Morphismes K -orientés d’espaces de feuilles et fonctorialité en théorie de Kasparov (d’après une conjecture d’A. Connes). *Ann. Sci. École Norm. Sup. (4)*, 20(3):325–390, 1987.
- [Ler10] Eugene Lerman. Orbifolds as stacks? *Enseign. Math. (2)*, 56(3–4):315–363, 2010.
- [Loo77] Eduard Looijenga. Root systems and elliptic curves. *Invent. Math.*, 38(1):17–32, 1976/77.
- [Lur] Jacob Lurie. Survey article on elliptic cohomology. (unpublished) <http://www.math.harvard.edu/~lurie/>.
- [May96] J. P. May. *Equivariant homotopy and cohomology theory*, volume 91 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996. With contributions by M. Cole, G. Comezana, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafyllou, and S. Waner.
- [McL79] John McLeod. The Kunneth formula in equivariant K -theory. In *Algebraic topology, Waterloo, 1978 (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1978)*, volume 741 of *Lecture Notes in Math.*, pages 316–333. Springer, Berlin, 1979.
- [PS86] Andrew Pressley and Graeme Segal. *Loop groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1986. Oxford Science Publications.
- [Qui71] Daniel Quillen. Elementary proofs of some results of cobordism theory using Steenrod operations. *Advances in Math.*, 7:29–56 (1971), 1971.

- [Ro01] Ioanid Roşu. Equivariant elliptic cohomology and rigidity. *Amer. J. Math.*, 123(4):647–677, 2001.
- [Ro03] Ioanid Roşu. Equivariant K -theory and equivariant cohomology. *Math. Z.*, 243(3):423–448, 2003. With an appendix by Allen Knutson and Roşu.
- [Seg68] Graeme Segal. Equivariant K -theory. *Inst. Hautes Études Sci. Publ. Math.*, (34):129–151, 1968.
- [Ser56] Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier, Grenoble*, 6:1–42, 1955–1956.
- [Tym09] Julianna Tymoczko. Divided difference operators for partial flag varieties. <http://front.math.ucdavis.edu/0912.2545>, December 2009.